CEU Mathematics Entry Test
2012

This test consists of two parts, spread over three pages, not including this one.

In Part I, there are 7 questions, each worth at least 2 points. No applicant with fewer than 7 points in Part I will be considered for admission.

In Part II, there are 10 questions for a total of 100 points. You should try to solve as many of them as possible. Use a separate page for answering each question. Be sure to explain and justify all your answers very carefully. Write legibly.

You have two and a half hours to complete the test.

Allocate your time carefully among the problems.

NO mobile phones, calculators, books, external materials or any other forms of assistance are allowed.

Good luck!
Part 1

**Problem 1 (2 points)** Prove that for all non-negative numbers \( x \) and \( y \)

\[
\sqrt{xy} \leq \frac{x + y}{2}.
\]

**Problem 2 (2 points)** Two workers working together finished a job in 12 hours. Later the first worker made one half of the same job and after it the second worker finished the other half of the job. In this case they finished the job in 25 hours. How long can they finish the job separately?

**Problem 3 (2 points)** Simplify the following expression

\[
\frac{x + \sqrt{x^2 - 4x}}{x - \sqrt{x^2 - 4x}} - \frac{x - \sqrt{x^2 - 4x}}{x + \sqrt{x^2 - 4x}}
\]

**Problem 4 (2 points)** Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). How much is \((A+2B)^{-1}\)?

**Problem 5 (2 points)** Let \( z = x^2 \ln y^3 \). Calculate the partial derivatives \( \partial z / \partial x \) and \( \partial z / \partial y \)

**Problem 6 (2 points)** Calculate the radius of convergence and the sum of the following series

\[1 + x^3 + x^6 + x^9 + \ldots\]

**Problem 7 (2 points)** Calculate the integral \( \int t \cos t \, dt \).
Part II

Problem 8 (6 points) Show that the vectors
\[
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]
are linearly independent.

Problem 9 (8 points) For which values of the parameter \(a\) the function
\[
f(x, y) = -6x^2 + (2a + 4)xy - y^2 + 4ay
\]
is concave?

Problem 10 (8 points) For which value of the parameter \(A\) the function
\[
f(x) = \begin{cases} Ax^3 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}
\]
is a density function of a random variable \(\xi\)? How much is the standard deviation of the variable \(1/\xi\)?

Problem 11 (8 points) Assume that \(P(A | B) = 0.7, P(A | B^c) = 0.3 \text{ and } P(B | A) = 0.6\). How much is \(P(A)\)?

Problem 12 (8 points) Calculate the following integrals!
1. \(\int_0^{\pi/2} \cos^3 x \, dx\)
2. \(\int \frac{x^3}{x^2 + 5} \, dx\).

Problem 13 (10 points) Prove that for every \(x > 0\)
\[
\frac{x}{1 + x} < \ln(1 + x).
\]

Problem 14 (12 points) For each of the following functions defined on \(\mathbb{R}^2\), find the critical points and classify them as local max, local min, saddle point, or "cannot tell" out in full:
1. \(z = x^3 + y^3 - 3xy\).
2. \(z = y^2x - 3xy + 2x^4\).

Problem 15 (12 points) Using the implicit differentiation rule calculate \(y'\) if
\[
2x^2 + 6xy + y^2 = 18.
\]
Problem 16 (14 points)  Find the conditional extrema of \( y^2 / (x^2 - 36) \) over \( x^2 - xy - 54 = 0 \). Try to use Lagrange-multipliers!

Problem 17 (14 points)  As a function of parameters \( a \) and \( b \) give the number of solutions of the linear equation

\[
\begin{align*}
x + 2y + 3z &= 1 \\
-x + ay - 21z &= 2 \\
3x + 7y + az &= b
\end{align*}
\]